Universal Scaling in Saddle-Node Bifurcation Cascades (I)

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Abstract

A saddle-node bifurcation cascade is studied in the logistic equation, whose bifurcation points follow an expression formally identical to the one given by Feigenbaum for period doubling cascade. The Feigenbaum equation is generalized because it rules several objects, which do not have to be orbits. The outcome is that an attractor of attractors appears, and information about the birth, death and scaling of windows is obtained.

Key words: Saddle-Node bifurcation cascade. Attractor of attractors. Generalized Feigenbaum equation. Scaling windows. Scaling Myrberg-Feigenbaum points. Bifurcation rigidity.

1 Introduction

The study of the logistic equation

$$x_{n+1} = f(x_n) = rx_n(1 - x_n) \quad 0 \le r \le 4 \tag{1}$$

bears relevance because it is a simple expression with a numerical and theoretical easy use that at the same time shows universal and varied behaviors. It is enough to mention the patterns of periodic orbit, found by Metropolis, Stein and Stein [1], or the Feigenbaum cascade [2,3], the accumulating point of which was discovered by Myrberg [4].

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As all quadratic maps are topologically conjugate [5] we can just focus the study in one of them. On the other hand, the iterated one-dimensional maps, under rather general conditions, are nearly quadratic if they are renormalized [6]. As a consequence, the work done with the logistic equation can be generalized to other maps. Therefore, any progress that makes clear the hierarchies of bifurcations and the structure of this equation helps to the extension of our knowledge of dynamical systems and the understanding of other associated phenomena.

Within the study of the logistic equation the identification and organization of orbits is outstanding, that is, its organization and hierarchies in more detailed structures which are self-similar. It is also relevant to establish in which order they are created, and from which kind of bifurcation, as well as, for which parameter values the orbits prevail [7,8,9,10]. In short, it is a question of establishing the structure of periodic windows, a question to which we want to contribute.

The study of the logistic equation is also important in Physics, both as a model [11] and because it helps to calculate magnitudes which describe certain physics processes: Lyapunov exponents and topological entropy. From an experimental point of view the spotting stable orbit would allow the experimenter to locate himself in a stable region, which is close to phenomena that he wants to observe: bifurcations and several kinds of chaos. On the other hand, the periodic orbits allow us a semiclassical approach of quantum mechanics [12]. Therefore, the progress in the understanding, in its mathematical aspect, of periodic orbits helps to the theoretical and experimental development of Physics.

Lately, an area of the study of application of logistic equation has focused on the distribution of periodic windows, their widths, their locations and its bifurcation diagram [13,14], clearly linked to central question, above mentioned, of establishing which is the complex structure of those windows. Similarly, we want to contribute with our work, in which we will show a saddle-node bifurcation cascade. This is a sequence of saddle-node bifurcations (tangent bifurcation), with a recurrence law identical to the one Feigenbaum found for period doubling cascade, which we will allow us to a) calculate the parameter values where the windows are born and die, and therefore their length, b) the identification and organization of orbits in windows, c) spot the accumulating point of a saddle-node bifurcation cascade, in which there will be chaos.

The works by Feigenbaum [2,3,15] and Hunt et al. [14] are the cornerstones to get the above mentioned points. The first one will be fundamental in the numerical, geometrical and theoretical interpretation of the results, meanwhile the second one will be so for the theoretical reconstruction of the behavior of the logistic equation as shown in its numerical analysis.

This paper is organized as follows:

In section II numerical results are shown. These results suggest that in an arbitrary period-j window, the birth of successive period-$q*2^n$ saddle-node orbits scale in the same way as in a Feigenbaum cascade. In general, inside the same period-j window, there will be several saddle-node bifurcation cascades with the same basic period q.

In section III we get the theoretical explanation of these properties. In the process we will find that not only the period doubling cascade and saddle-node bifurcation cascade scale in the same way but also that it is typical of other sequences which appear in the logistic equation, such as accumulating points of Myrberg-Feigenbaum points.

In section IV, and following the work by Feigenbaum for the period doubling cascade, we develop a similar work for saddle-node cascade which ends up in Feigenbaum-Cvitanoviç equation. It allows to explain the self-similarity of iterated functions and to generalize the results to polynomials with a maximum other than quadratic.

The results of section IV allow to generalize the bifurcation rigidity Principle [14] which is done in section V.

Section VI shows a rule to get the symbolic sequences of the saddle-node bifurcation cascade orbits, as well as a rule to order some of them.

In the conclusion section, some connections with a possible physical model are shown, which will most likely make the phenomenon fit in the framework described in this work.

2 Numerical results

In the logistic map the period-2^n orbits stem from pitchfork bifurcation cascade: Feigenbaum cascade. However, the period-$q = 3, 5, 7, ..., 2n + 1, ..$ orbits stem from saddle-node bifurcations (it is also possible to find saddle-orbits in which q is even, but not power of two).

Let f^q be the q-th iterated of f. When f^q, f obtained from Eq. (1), has a saddle-node bifurcation q saddle-node fixed points are generated at the same time. The node and saddle points pull away from each other when the control parameter is varied. Every q node point stands for an orbit, and each orbit will be, in turn, subjected to a Feigenbaum cascade. As result of that period-$q*2^n$ orbits are born. However, there are period-$q*2^n$ orbits which do not

stem from a period-q orbit displaying pitchfork bifurcation, but from f^{q*2^n} displaying saddle-node bifurcations.

Whereas period-$q*2^n$ orbits duplicate their period because of Feigenbaum cascade as the parameter r is increased in Eq. (1), contrary to what happens with saddle-node orbits. So, there is a period-3 saddle-node bifurcation at $r_3 = 1+\sqrt{8}$ [16] (see Fig. 1). If r is increased, to get the point of pitchfork bifurcation, a period-$3*2$ orbit is generated. A further increase will generate a new pitchfork bifurcation and a period-$3*2^2$ orbit. A Feigenbaum cascade is taking place inside a period-3 window

On the contrary if we decrease r from $r_3 = 1+\sqrt{8}$ the period-3 orbit vanishes thus generating a saddle-node-$3*2$ period at $r_{3*2} \cong 3.6265$ (see Fig. 2). If r is decreased further the saddle-node period-$3*2$ orbit vanishes, and so a saddle-node period-$3*2^2$orbit is generated at $r_{3*2^2} \cong 3.5820$ (see Fig. 3). A period-$3, 3*2, 3*2^2, .., 3*2^n, ...$ saddle-node bifurcation cascade takes place. Refer to Table 1 for several birth saddle-node orbit values. We can see that this saddle-node bifurcation cascade is taking place at a canonical window, whereas pitchfork bifurcations are taking place at a period-3window.

Following Feigenbaum [2] let's define

$$\delta_{3*2^n} = \frac{r_{3*2^{n+1}} - r_{3*2^n}}{r_{3*2^{n+2}} - r_{3*2^{n+1}}}$$

and

$$\alpha_{3*2^n} = -\frac{d_{3*2^n}}{d_{3*2^{n+1}}}$$

where r_{3*2^n} is the value of r at which a period-$3*2^n$ orbit is created. And for this same saddle-node orbit we define d_{3*2^n} as the distance from $x = \frac{1}{2}$ to the nearest saddle-node point.

In Table 1 estimates of α_n and δ_n for several values of n are shown, which apparently approach the $\delta = 4.66920160..$ and $\alpha = 2.50290787...$ Feigenbaum constants. Similar results are reached if we repeat the process for the saddle-node period-$5, 5*2, 5*2^2, .., 5*2^n, ..$ orbit, as shown in Table 2.

The former two cases relate to saddle-node bifurcation cascade at a canonical window, but the same kind of results are obtained working with other windows. For instance, inside a period-3 window a saddle-node bifurcation cascade can be generated, that is, period-$3, 3*2, 3*2^2, .., 3*2^n, ..$ orbits are created at this window. Nevertheless, it relates to period $3*3, 3*3*2, 3*3*2^2, .., 3*3*2^n, ..$ orbits for the original map Eq. (1). If this same cascade was generated at period-5window period-$5*3, 5*3*2, 5*3*2^2, .., 5*3*2^n, ..$ orbits would

be created at a canonical window. In both cases, as shown in Tables 3 and 4, again we have the same convergence to the δ Feigenbaum constant.

Let's emphasize that the saddle-node period-5∗3 orbit is a saddle-node period-3 orbit which displays at a period-5window; different from a saddle-node period-15 displayed at a canonical window, and different from a saddle-node period-3∗5 orbit, as well, the latter being born as a saddle-node period-5 orbit at a period-3 window .

As well as numerical similarity with Feigenbaum's work, there is another geometrical one. As shown in Figs. 1, 2 and 3 the graph of f^3(Fig. 1) is replicated in the neighborhood of $(\frac{1}{2}, \frac{1}{2})$, for any orbit of the period-$3, 3*2, 3*2^2, .., 3*2^n, ...$ saddle-node bifurcation cascade. The same phenomenon can be noticed in other critical points of f^{2^n}close to the line $y = x$. These geometrical results apply not only to a $3, 3*2, 3*2^2, .., 3*2^n, ..$ cascade, but also to $q, q*2, q*2^2, .., q*2^n, ..$ $q \neq 2^m$ cascades.

The geometrical, as well as the numerical, phenomenon apply not only to a canonical window, as shown in Fig. 4, where the $3, 3*2, 3*2^2, .., 3*2^n, ..$ cascade is at the period-5window.

Summarizing, what the numerical results suggest is that for a given saddle-node bifurcation, where a period-$q \neq 2^m$ orbit is created, there is a saddle-node bifurcation cascade where period-$q*2^n$ orbits are created. Let r_{q*2^n} $n = 0, 1, 2, 3, ...$ be the value of r at which a period-$q*2^n$ saddle-node bifurcation is created. As r is decreased from r_q the different bifurcation values appear as

$$... < r_{q*2^n} < ... < r_{q*2} < r_q$$

resulting in

$$\delta = \lim_{n\to\infty} \delta_{q*2^n} = \lim_{n\to\infty} \frac{r_{q*2^{n+1}} - r_{q*2^n}}{r_{q*2^{n+2}} - r_{q*2^{n+1}}} \tag{2}$$

and

$$\alpha = \lim_{n\to\infty} \alpha_{q*2^n} = \lim_{n\to\infty} \left(-\frac{d_{q*2^n}}{d_{q*2^{n+1}}}\right) \tag{3}$$

where α, δ are Feigenbaum's constants, and d_{q*2^n} the distance from $x = \frac{1}{2}$ to the nearest saddle-point at $r = r_{q*2^n}$.

As well as in the Feigenbaum cascade, when three consecutive values $r_{q*2^{bin-3}}, r_{q*2^{n-2}}, r_{q*2^{n-1}}$are given the following bifurcation value r_{q*2^n} can be approximately predicted.

This result is important because the values where saddle-node bifurcation appear are the values where period-$q * 2^n$ windows are born. Therefore, we can spot the birth of period-$q * 2^n$ windows, once we have found three consecutive bifurcation values. In addition, it is possible to know how the lower endpoints of such periodic windows scale.

The Eq. (2) and (3) allow us to calculate what it is happening in the non-primary windows, because each period-$q * 2^n$ window, created in a saddle-node bifurcation, necessarily mimics the canonical window.

3 Generalization of Feigenbaum's formulas

3.1 Saddle-Node bifurcation cascade

Following Feigenbaum work, let be the sequence

$$r_1, r_2, r_3, ..., r_n..... \tag{4}$$

made up by values of r, for which Eq. (1) shows a pitchfork bifurcation. If the sequence is ordered in such a way that a period-$j * 2^n$ orbit is created at r_n, being j the original orbit period, according to Feigenbaum [2] there exist the limits

$$\delta = \lim_{n \to \infty} \frac{r_{n+1} - r_n}{r_{n+2} - r_{n+1}} = \lim_{n \to \infty} \delta_n \tag{5}$$

$$r_\infty = \lim_{n \to \infty} r_n$$

From the former limits, the expression

$$r_{n+1} = \frac{1}{\delta} r_n + (1 - \frac{1}{\delta}) r_\infty \tag{6}$$

follows, valid for $n > N$, being N big enough.

For further development it is necessary to write Eq. (6) in terms of the bounds of an interval and we can do so by using a period-j window originating from a saddle-node bifurcation. $r = r_{j,ini}$ is the lower bound of the window and $r = r_{j,end}$ the upper bound. The window has a Myrberg-Feigenbaum point $r_{\infty,j}$, where a period-$j * 2^n$cascade finishes. Then, as Eq. (6) there is an $r_{N,j}$

so that for every $r_{n,j}$ accomplishing $r_{N,j} \leq r_{n,j} \leq r_{\infty,j}$ the equation

$$r_{n+1,j} = \frac{1}{\delta} r_{n,j} + (1 - \frac{1}{\delta}) r_{\infty,j} \qquad (7)$$

is accomplished.

With this posing, the lineal Eq. (6) turns to lineal Eq. (7), which is accomplished in the subset $[r_{n,j}; r_{\infty,j}] \subset [r_{j,ini}; r_{j,end}]$.

We know that bifurcation points $r_{n,j}$ of doubling-period cascade scale as Eq. (7). We are going to prove that bifurcation points of saddle-node bifurcation cascade, which have been encountered numerically, escalate as Eq. (7) too, that is

$$r_{q*2^{n+1},SN,j} = \frac{1}{\delta} r_{q*2^n,SN,j} + (1 - \frac{1}{\delta}) r_{\infty,j}$$

at $r_{q*2^n,SN,j}$ the function f^j bifurcate to a saddle-node period-$q*2^n$ orbit.

To get this result the Eq. (7) must extend to interval $[r_{n,j}; r_{q*2^n,SN,j}]$. Accordingly let's take the interval $[r_{n,j}; r_{q*2^n,SN,j}] \subset [r_{j,ini}; r_{j,end}]$, which holds the interval $[r_{n,j}; r_{\infty,j}]$ where Eq. (7) is accomplished.

The key of proof is to chose two intervals $[r_{n,j}; r_{q*2^n,SN,j}]$ and$[r_{n+1,j}; r_{q*2^{n+1},SN,j}]$, whose bounds are bifurcation points of f^j. Both intervals will be related linearly. The Eq. (7) shows that the lower bound of an interval is linearly transformed in the lower bound of the other interval. By the uniqueness of linear transformation between both intervals, the upper bounds are linearly transformed by Eq. (7).

Starting from [17], Hunt *et al.* [14] demonstrated the following principle:.

Principle 1: Let X and Λ be compact intervals. For a typical C^3 family maps $f : X \times \Lambda \to X$ and a typical superstable period n orbit with critical point (x_0, λ_0), there is a linear change of coordinates that conjugates f^nnear (x_0, λ_0) to the quadratic family $y \longmapsto y^2 - c$ in the square $[-2.5, 2.5] \times [-2.5, 2.5]$ to within an error ε in the C^2 norm, where $\varepsilon \to 0$ as n increases.

Principle 1 shows that close to critical point the bifurcation diagram for a window can be linearly transformed into a canonical bifurcation diagram.

Let's see Fig. 4. The first element of the $3 * 2^m$ saddle-node cascade is shown. In the neighborhood of $x = \frac{1}{2}$ of Fig. 4 the graph of f^3 is seen (Fig. 1). The graph of f^5 has been superimposed, which has a parabolic shape in the neighborhood of $x = \frac{1}{2}$. If we expanded f^5 in the neighborhood of critical

point $x = \frac{1}{2}$ it would turn into a quadratic polynomial, which if iterated three times it would look like a graph of f^3 as shown in Fig. 4. The same happens for all five critical points of f^5, each one of which would have a graph of f^3 tangent to line $y = x$,and the result would be a period-$3 * 5 = 15$ orbit in the canonical window.

Principle 1 suggests that the bifurcation diagram (of quadratic approximation of f^5 close to critical point) is linearly transformed into canonical bifurcation diagram. This linear transformation turns value r , which accounts for graph of f^3in Fig. 4, into value r which accounts for graph of f^3 in Fig. 1.

As said before we formulate the following principle:

Principle.- In a period-j window, saddle-node bifurcation cascades scale by

$$r_{q*2^{n+1},SN,j} = \frac{1}{\delta} r_{q*2^n,SN,j} + (1 - \frac{1}{\delta}) r_{\infty,j}$$

Proof.

If the bifurcation diagram of f^jin $[r_{n,j}; r_{q*2^n,SN,j}]$ and the f^{2*j}in $[r_{n,2*j}; r_{q*2^n,SN,2*j}]$ are considered, according to Principle 1 there will be

i) A linear transformation that maps bifurcation diagram of f^j in $[r_{n,j}; r_{q*2^n,SN,j}]$ onto canonical bifurcation diagram of f in $[r_{n,1}; r_{q*2^n,SN,1}]$, let this be h_j. At $r_{n,1}$ the nth pitchfork bifurcation of f occurs. At $r_{q*2^n,SN,1}$a period-$q * 2^n$ saddle-node bifurcation of f occurs.

ii) a linear transformation that maps bifurcation diagram of f^{2*j}in $[r_{n,2*j}; r_{q*2^n,SN,2*j}]$ onto bifurcation diagram of f in $[r_{n,1}; r_{q*2^n,SN,1}]$, let this be h_{2j}.

Therefore, in the parameter space, there will be a linear transformation $h_{2j}^{-1} \circ h_j$ that maps the bifurcation diagram of f^j in $[r_{n,j}; r_{q*2^n,SN,j}]$ onto bifurcation diagram of f^{2*j} in $[r_{n,2*j}; r_{q*2^n,SN,2*j}]$. As the bifurcation diagram of f^{2*j}in $[r_{n,2*j}; r_{q*2^n,SN,2*j}]$ coincides with bifurcation diagram of f^j $[r_{n+1,j}; r_{q*2^{n+1},SN,j}]$, the result is that the linear transformation $h_{2j}^{-1} \circ h_j$ maps the bifurcation diagram of f^j of $[r_{n,j}; r_{q*2^n,SN,j}]$ onto the bifurcation diagram of f^jin $[r_{n+1,j}; r_{q*2^{n+1},SN,j}]$.

Let's notice how Eq. (7) is a linear transformation that maps the interval $[r_{n,j}; r_{\infty,j}]$ onto $[r_{n+1,j}; r_{\infty,j}]$. As the linear transformation is unique, the linear transformation $h_{2j}^{-1} \circ h_j$ within the limits of $[r_{n,j}; r_{\infty,j}] \subset [r_{n,j}; r_{q*2^n,SN,j}]$ must coincide with Eq. (7), i.e., $r_{n+1,j} = h_{2j}^{-1} \circ h_j(r_{n,j}) = \frac{1}{\delta} r_{n,j} + (1 - \frac{1}{\delta}) r_{\infty,j}$. However, the linear transformation is accomplished on the whole interval $[r_{n,j}; r_{q*2^n,SN,j}]$and not only on the $[r_{n,j}; r_{\infty,j}] \subset [r_{n,j}; r_{q*2^n,SN,j}]$ where a pitchfork bifurcation cascade occurs. By applying the same lineal transformation

to upper bound of interval the expression

$$r_{q*2^{n+1},SN,j} = \frac{1}{\delta} r_{q*2^n,SN,j} + (1 - \frac{1}{\delta}) r_{\infty,j} \tag{8}$$

results.

For that reason it is demonstrated that bifurcation parameter values in saddle-node bifurcation cascade are governed by the same relation that governs Feigenbaum cascade as shown in Eq. (6). Accordingly, the convergence on $\delta$of Feigenbaum

$$\delta = \lim_{n\to\infty} \delta_{q*2^n} = \lim_{n\to\infty} \frac{r_{q*2^{n+1}} - r_{q*2^n}}{r_{q*2^{n+2}} - r_{q*2^{n+1}}}$$

suggested by numerical results is satisfied.

Obviously a saddle-node bifurcation cascade in a canonical window escalates in the same way as in a period-j window, as a result of the linear relation between the bifurcation diagram of one window and the other according to Principle 1.

3.2 Non-uniqueness of Saddle-Node bifurcation cascade

Most of the times, f^j will have saddle-node bifurcations not only for a single value of r but for several (although f^3 has only one saddle-node bifurcation). Therefore, the interval $[r_{n,j}; r_{q*2^n,SN,j}]$ used to obtain the Eq. (8) will not be unique, as there will be different values $r_{q*2^n,SN,j}$, with fixed q, n, j, for which f^j has saddle-node bifurcation. That means, there will be different values $r_{q*2^n,SN,j}$, with fixed q, n, j, accomplishing the same Eq. (8). Therefore, the saddle-node bifurcation cascade that escalates as Eq. (8) will not be unique either. Although saddle-node orbits will have the same period $q * 2^n$.

For instance, f^6 shows three different $6 * 2^n$ saddle-node bifurcation cascades, and none of them is related with the $3 * 2^{n+1}$ saddle-node cascade, although all of them have orbits with the same periods. As shown in Table 1, it is obvious that f^6 has a saddle-node bifurcation at $r \cong 3.6265$, but it must not be understood that a $6 * 2^n$ saddle-node bifurcation cascade starts at this point, but it must be considered as the second element of the $3 * 2^n$ cascade, since the graph of f^3 appears in the neighborhood of the critical points of f^2nearest to line $y = x$ (see Fig. 2). The following element of the $3 * 2^n$cascade occurs at $r \cong 3.5820$ (see Fig. 3), where the same figure is repeated 2^2 times. However, f^6 has saddle-node bifurcation at $r_1 \cong 3.9375$, $r_2 \cong 3.9779$ and

$r_3 \cong 3.9976$, and it is impossible for f^3 to have saddle-node bifurcation at these points. In each of all three former points a $6 * 2^n$ saddle-node cascade starts. The following element of the respective cascades must reproduce the graph of f^6 and duplicate its period, which occur at $r_a \cong 3.6552$, $r_b \cong 3.6684$ and $r_c \cong 3.6767$ respectively. Obviously, for these three new values f^{12} has a saddle-node bifurcation, but as said before, it must not be understood as the beginning of a $12 * 2^n$ cascade, because the graph of f^6 is repeated twice. In Fig. 6 and 7 two elements of $6*2^n$ cascade are shown, corresponding to r_2 and r_b.

3.3 Other objects scaling with Feigenbaum's relation

The reasoning followed to show the saddle-node bifurcation cascade scaling can be used with other "objects" . To do so, such "objects" must belong to a sequence, and later build intervals with the elements of the sequence in a similar way to the proof of a saddle-node bifurcation cascade.

We can probe it with Myrberg-Feigenbaum points. To do so, let us consider the period-$q * 2^n$window $n = 0, 1, 2, ..$, born of a primary saddle-node orbit. Each window will have a Myrberg-Feigenbaum point $r_{\infty,q*2^n}$, where the Feigenbaum cascade finishes. Let us build the sequence

$$r_{\infty,q,j}, r_{\infty,q*2,j}, ..., r_{\infty,q*2^n,j} \tag{9}$$

that represents the parameter values of Eq. (1), in which every one is the end of a Feigenbaum cascade in period-$q, q * 2, q * 2^2, .., q * 2^n, ..$ windows respectively, that is , Myrberg-Feigenbaum points. Period-$q, q * 2, q * 2^2, .., q * 2^n, ..$ windows are inside period-j window, because of this the subindex j in Eq. (9). This sequence is vital for the proof.

In proof of the saddle-node cascade convergence let $[r_{n,j}; r_{q*2^n,SN,j}]$ and $[r_{n,2*j}; r_{q*2^n,SN,2*j}]$ be substituted by$[r_{n,j}; r_{\infty,q*2^n,j}]$ and $[r_{n,2*j}; r_{\infty,q*2^n,2*j}]$. Initially $r_{q*2^n,SN,j}$ used to indicate the value of parameter for which f^j had a period-$q*2^n$ saddle-node orbit. Such orbit will have a Myrberg-Feigenbaum at some value of r, which we represent by $r_{\infty,q*2^n,j}$. This is the value that appears in the new intervals, and also in the sequence (9).

In order to demonstrate that the Myrberg-Feigenbaum points are governed by

$$r_{\infty,q*2^{n+1},j} = \frac{1}{\delta} r_{\infty,q*2^n,j} + (1 - \frac{1}{\delta}) r_{\infty,j} \tag{10}$$

$n > N$, being N large enough, we simply need to rebuild the saddle-node cascade demonstration in the initial way, but with new intervals.

Let's expound what Eq. (10) means. There will be saddle-node period-$q*2^n$orbits inside a period-j window born of a period-j saddle-node orbit. After the birth of these orbits at $r_{q*2^n,SN,j}$a Feigenbaum cascade occurs, which will finish at Myrberg-Feigenbaum point $r_{\infty,q*2^n,j}$. Then, the Eq. (8), which interrelates birth-point of period-$q*2^n$ orbits, is identical to the equation which interrelates the end-points where doubling-period cascades of those same orbits finish.

The Eq. (10) brings more information about Myrberg-Feigenbaum points. It is well known that the Myrberg-Feigenbaum point $r_{\infty,j}$ is the accumulation point of values r where pitchfork bifurcations occur, because a doubling-period cascade finishes at $r_{\infty,j}$. However, the Myrberg-Feigenbaum point $r_{\infty,j}$is not only the accumulation point of Feigenbaum cascade because the Eq. (10) shows that $r_{\infty,j}$ is the accumulation point of Myrberg-Feigenbaum points $r_{\infty,q*2^n,j}$,that is, Myrberg-Feigenbaum point $r_{\infty,j}$ is the accumulation point of Myrberg-Feigenbaum points of the period-$q*2^n$ $q\neq 2^m$ primary windows, which have been born inside period-j windows. As Myrberg-Feigenbaum points are attractors what we are in front of is an attractor of attractors of attractors of ...

The problem of the convergence has not yet been completed. Let's notice that at $r_{\infty,j}$ not only a single sequence of Myrberg-Feigenbaum points convergences, but infinite ones, because for each $q\neq 2^m$ there is an associated sequence $q*2^n$ $n=0,1,2,3,...$, and for each sequence the same happens. The whole process is repeated for each value of $q\neq 2^m$.

The reasoning followed with Myrberg-Feigenbaum points can be applied to other "objects". The reader cannot miss that if instead of choosing Myrberg-Feigenbaum points $r_{\infty,q*2^n,j}$, where a Feigenbaum cascade finishes, we choose the point $r_{q*2^n,end,j}$ where the window started at $r_{q*2^n,SN,j}$finishes then equation

$$r_{q*2^{n+1}end,j}=\frac{1}{\delta}r_{q*2^n,end,j}+(1-\frac{1}{\delta})r_{\infty,j} \tag{11}$$

$n>N$, being N large enough, will be the same.

Notice that three demonstration have been carried out for three different points of a same window respectively, that is, $r_{q*2^n,SN,j}$, $r_{\infty,q*2^n,j}$ and $r_{q*2^n,end,j}$, initial point, intermediate point and final point.

From the equations found we can answer part of the questions initially posed: "where windows begin and end", the answers to which are given by Eq. (8) and (11). From these results the relation between the length $L_n$of the period-$q*2^n$

windows is obtained, because by subtracting Eq. (8) and (11)

$$L_{n+1=}r_{q*2^{n+1},end,,j} - r_{q*2^{n+1},SN,j} = \frac{1}{\delta}r_{q*2^n,end,,j} - \frac{1}{\delta}r_{q*2^n,SN,j} = \frac{1}{\delta}L_n$$

is obtained, that is,

$$\delta = \frac{L_n}{L_{n+1}} \tag{12}$$

This equation simply means that the quotient of the length of two intervals which are linearly transformed one onto other must be constant. The constant δ must be the proportional factor in the lineal map, or else, how the lengths of the two consecutive windows born of saddle-node bifurcation cascade are contracted. For the same reason, the equation

$$\delta = \frac{r_{q*2^n,end,j} - r_{q*2^n,M,j}}{r_{q*2^{n+1},end,j} - r_{q*2^{n+1},M,j}} \tag{13}$$

applies.

Let's bear in mind that we are talking about period-$q * 2^n$ windows, which scales in the same period-j window, and not about how a period-j window scales with respect to a canonical window. Therefore there is no conflict with other theoretical results [13,17].

3.4 Generalized Feigenbaum's relation

Eqs. (8), (10) and (11) are identical to the Eq. (6), and they are valid for $r > r_\infty$, meanwhile Eq. (6) is valid for $r < r_\infty$. We can express all these results in only one equation and generalize the Feigenbaum equation (Eq. (6)) , by writing

$$r_{n+1,j} = \frac{1}{\delta}r_{n,j} + (1 - \frac{1}{\delta})r_{\infty,j} \tag{14}$$

where points $r_{n,j}$ show the elements of a sequence (pitchfork bifurcation, saddle bifurcation, Myrberg-Feigenbaum point, supercycle or any other element to which the proof of saddle-node bifurcation applies).

The Eq. (14), apart from being a generalization, informs us about the order in which the different elements of the sequence appear. The order will depend on the parameter value $r_{n,j}$ being smaller or bigger than $r_{\infty,j}$.

From the Eq. (14) we can deduce that if $r_{n,j} \prec r_{\infty,j}$ then $r_{n,j} \prec r_{n+1,j} \prec r_{\infty,j}$, on the contrary if $r_{n,j} \succ r_{\infty,j}$ then $r_{\infty,j} \prec r_{n+1,j} \prec r_{n,j}$. From which we deduce that if a bifurcation occurs for a value such that $r_{n,j} \prec r_{\infty,j}$ then the next value of bifurcation increases, in the same way as for pitchfork bifurcation. On the contrary, when the bifurcation occurs for $r_{n,j} \succ r_{\infty,j}$ then the next values of bifurcation decreases, in the same way as for saddle-node bifurcation cascades.

In this ordering, part of these results are new. If we consider first-occurrence orbits the former results turn out to be a particular case of a theorem (Theorem 2.10 in [8]). However, the ordering shown in this section deals with period-j window, which does not have to originate from first-occurrence period-j windows. For this reason, the ordering is not restricted to first-occurrence orbits and therefore it is not regarded in the Sharkovsky theorem.

On the other hand, the ordering that we have just shown is not restricted only to orbits, as the Sharkovsky theorem, but it also applies to Myrberg-Feigenbaum points and to other objects that are not orbits. An ordering which was hidden as far as we know.

Therefore the Eq. (14) enlarges the ordering to both new orbits and other objects which are not orbits.

Let's summarize the conclusions obtained in this section for a period-$j$window:.

i) It is possible to spot the birth of a saddle-node period-$q*2^n$ orbit and therefore to fix the birth of period-$q*2^n$ windows from Eq. (8). This equation justifies the numerical results shown in Tables 1, 2, 3 and 4, where the convergence to Feigenbaum δconstant is shown.

ii) Equally it spots where a period-$q*2^n$ window finishes from Eq. (11)

iii) According to the former two points the scaling of windows related to saddle-node bifurcation cascades is proved Eq. (12).

iv) The Feigenbaum equation (Eq. (6)) has been generalized, both in the range of the parameter validity and in the objects to which is applied

v) In particular, the generalization of Eq. (6) is applied to Myrberg-Feigenbaum points, showing that these points behave as an attractor of attractors.

vi) The generalization, applied to Myrberg-Feigenbaum points, allows us to spot where the Feigenbaum cascade of period-$q*2^n$ window finishes by means of Eq. (10), or else, it shows the relative position of the attractors in which doubling-period cascades finish.

vii) The ordering of the objects governed by Eq. (14) is shown, and if they happen to be orbits they do not have to be first-occurrence orbits.

In this way, we account for the question initially posed, about the birth, end, scaling and structure of the orbits.

4 Self-similarity and convergence to Feigenbaum's α constant

4.1 Self-similarity

In the numerical results we anticipated that the graph of f^3(Fig. 1), when it has a saddle-node bifurcation, is repeated in the neighborhood of point $(\frac{1}{2}, \frac{1}{2})$ both $f^{3\cdot 2}$ and $f^{3\cdot 2^2}$, when $f^{3\cdot 2}$ and $f^{3\cdot 2^2}$have also saddle-node bifurcation. Furthermore, the graph of f^3 in the neighborhood of point $(\frac{1}{2}, \frac{1}{2})$ appears contracted by a factor $\frac{1}{\alpha}$, each time $f^3$is iterated with itself, where $\alpha$is a Feigenbaum constant.

Let's justify the numerical results, as well as why similar figures are shown at the 2^n critical points of f^{2^n}near line $y = x$ (see Fig. 3) These similar figures are like saddle-node orbits trapped in the critical points, and we will use them for the mathematical analysis.

A saddle-node period-$q*2^n$ orbit is obtained when the derivative of f^{q*2^n} takes value 1. This orbit is forced to a period doubling process, a process that will occur when the derivative takes value -1. Accordingly, the derivative vanishes in some middle point, which corresponds to a supercycle. Therefore, we can associate each saddle-node orbit with a supercycle.

Feigenbaum [2] introduced the quotient

$$\delta_n = \frac{R_{n+1} - R_n}{R_{n+2} - R_{n+1}} \tag{15}$$

where $R_n$is the value for which Eq. (1) has a period-2^{n+1} supercycle in a period doubling cascade. The quotient (Eq. (15)) approaches aδ Feigenbaum constant.

Let's follow Feigenbaum and define

$$\delta_n = \frac{r_{n+1} - r_n}{r_{n+2} - r_{n+1}} \tag{16}$$

being $r_n$the value where the period-$q * 2^n$ supercycle occurs. Such supercycle

is associated to a period-$q*2^n$ saddle-node orbit. Then we will get the same results as Feigenbaum: the convergence of the quotient (Eq. (16)) on δ constant (see Table 1). This result leads us to the definition of a family of functions similar to those created by Feigenbaum in his work

As for Eqs. (15) and (16), we have to emphasize that we will work with supercycles associated to orbits born of saddle-node bifurcations instead of supercycles associated to orbits born of pitchfork bifurcation as Feigenbaum does. There is an important difference with regard to the behavior of both kinds of orbits in the parameter space. The orbits of period doubling cascade are adjacent in a connected set: an interval of the parameter space. However, the saddle-node period-$q*2^n$ orbits only exist in discrete points of the parameter space. Besides, saddle-node orbits that do not belong to the same cascade can be found between saddle-node period-$q*2^n$and period- $q*2^{n+1}$ orbits, which proves that the latter orbits are not adjacent. Let us mention other essential difference between both bifurcations: the pitchfork bifurcation has a local character, whereas the Saddle-Node bifurcation leads to a global qualitative change in the behavior of the logistic equation.

Bearing in mind the previous explanation and following Feigenbaum, we define (see Eq. (35) of [15])

$$g_{1,q} = \lim_{n\to\infty} (-\alpha)^n f^{2^n}_{R_{n+1,q}} \left[\frac{x}{(-\alpha)^n}\right]$$

where the supercycle associated to saddle-node period-$q*2^n$ orbit is obtained at $R_{n,q}$, and q is fixed.

From previous definition we define a family of functions (see Eq. (39) of [15])

$$g_{i,q} = \lim_{n\to\infty} (-\alpha)^n f^{2^n}_{R_{n+i,q}} \left[\frac{x}{(-\alpha)^n}\right] \tag{17}$$

where q is fixed, which fulfills the next equation (see Eq. (42) of [15])

$$g_{i-1,q} = (-\alpha)\, g_{i,q} \left[g_{i,q}\left(-\frac{x}{\alpha}\right)\right] \equiv T g_{i,q}(x) \tag{18}$$

Taking the limit in Eq. (17) (see Eq. (40) of [15]) the function converges to a limiting function

$$g(x) = \lim_{i\to\infty} g_{i,q} \tag{19}$$

This limit exists at a fixed value of r, the Myrberg-Feigenbaum r_∞, and

it means that there is an accumulation point at r_∞ for a saddle-node $q * 2^n$bifurcation cascade.

It follows from Eqs. (18) and (19) that

$$g(x) = -\alpha g\left[g\left(-\frac{x}{\alpha}\right)\right] \tag{20}$$

(see Eq. (43) of [15])

It follows from Eq. (20) that if $g(x)$ is the solution of this equation then $g^q(x)$ is the solution of

$$g^q(x) = -\alpha g^q\left[g^q(-\frac{x}{\alpha})\right] \tag{21}$$

This equation means that the iterated of g^q is self-similarly scaled by α. Precisely, this is shown in Fig. 1, 2 and 3, in the neighborhood of $\left(\frac{1}{2},\frac{1}{2}\right)$: the graph of f^3, close to a saddle-node bifurcation, is repeated scaled by a constant. This constant is the nearer to α the bigger n is in f^{3*2^n}, if f^{3*2^n} has a saddle-node bifurcation (see Table 1). With this last reasoning all numerical results shown in section 2 have been verified.

The Eq. (20), that we have just found, is the Feigenbaum-Cvitanoviç equation, which Feigenbaum utilized to show why function f is repeated in the neighborhood of $(\frac{1}{2},\frac{1}{2})$ in a period doubling cascade. If so, we wonder, why the original work of Feigenbaum is not utilized to justify the self-similarity of f^3in a saddle-node bifurcation cascade. The reason lies in the fact that Feigenbaum reached Eq. (20) from a limit of functions associated with pitchfork bifurcations, and accordingly its validity is not justified when we work with functions associated with other kinds of bifurcations, as is the case with saddle-node bifurcations.

This is not the only difference. It has been said that Eq. (20) is the limit of Eq. (18) only at $r = r_\infty$, and Feigenbaum got this value on the left, i.e., with $r < r_\infty$ which is how a period doubling cascade occurs. However we have got r_∞ on the right, as saddle-node bifurcations occur for $r > r_\infty$. That is, the Eq. (20) can be reached by limits of functions defined for both $r < r_\infty$and $r > r_\infty$, and that explains why Eq. (14) generalizes Eq. (6) of Feigenbaum, as we will see later.

4.2 Derivation of the generalized Feigenbaum formula.

Feigenbaum [3] got the Eq. (6) from the equation

$$g_{i+1}(x) - g_i(x) = \delta^{-i} h(x) \quad i \gg 1 \tag{22}$$

where $g_i(x)$ is the family of functions that we have utilized to define the family $g_{i,q}$ given by Eq. (17). $g_i(x)$ and $g_{i,q}$ are formally identical, but $g_i(x)$ refers to pitchfork bifurcations and $g_{i,q}$ refers to saddle-node bifurcations. In order to define well the Eq. (22) it is necessary to point out what $h(x)$ and δ are, which are calculated from

$$\delta h(x) = L_g h(x) \tag{23}$$

where

$$L_g h(x) = -\alpha \left\{ g' \left[g(\frac{-x}{\alpha}) \right] h(\frac{-x}{\alpha}) + h \left[g(\frac{-x}{\alpha}) \right] \right\} \tag{24}$$

where α and $g(x)$ are given by the Eq. (20).

If we follow Feigenbaum, working with $g_{i,q}$ instead of g_i, we will calculate an equation identical to Eq. (22), i.e.,

$$g_{i+1,q}(x) - g_{i,q}(x) = \delta^{-i} h(x) \quad i \gg 1 \tag{25}$$

where $g_{i,q}$ is the family of functions given by Eq. (17), whereas both $h(x)$ and δ are given by Eqs. (23) and (24). As both $g_i(x)$ and $g_{i,q}$ are identical and both Eqs. (22) and (25), which govern the former functions, are also identical then the result has to be the same: the Eq. (6). Although, this time the equation refers to a saddle-node bifurcations for $r > r_\infty$instead of a pitchfork bifurcation cascade for $r < r_\infty$, as Feigenbaum reached, that is, the Eq. (6) is valid for both $r > r_\infty$ and $r < r_\infty$. The thing is that depending on the fact that wether r is smaller or bigger than r_∞, the equation is assigned to one bifurcation or another.

The result of the former explanation is that Eq. (14), which generalizes Eq. (6), is obtained without the bifurcation rigidity Principle of Hunt et al. [14], as it was done in section 3.1, and it will be the starting point to generalize the bifurcation rigidity Principle in section 5.

Now we have obtained the equation which governs bifurcation points in saddle-node bifurcation cascades. We must distinguish two cases: saddle-node bifur-

cation cascades with the same or different basic periods. In a $q, q*2, q*2^2, .., q*2^n, ..$ $q \neq 2^m$ saddle-node bifurcation cascade q will be named a basic period.

4.2.1 Saddle-Node bifurcation cascades with the same basic period

As shown in section 3, there are different saddle-node bifurcation cascades with the same basic period in the canonical window, which are generated because the same period basic q is produced for different values of parameter r. Accordingly, there are different values $R_{n,q}$, with fixed q and n, and therefore the family of functions given by Eq. (17) are not the only ones, but there are as many families as there are values of $R_{n,q}$ (see sections 3.2 and 4.1). Given that there are different $g_{i,q}$families then there are different equations like Eq. (25). As Eq. (6) is obtained from Eq. (25) it turns out that there will be one Eq. (6) for each saddle-node bifurcation cascade. All equations will be identical and all of them will approach to r_∞, because the limit given by Eq. (20) exists only for r_∞ which does not depend on value of the $R_{n,q}$. What we have just found is something already known: for each fixed period q there are different saddle-node bifurcation cascades, each one of which is governed by the Eq. (6), and all of them approach to r_∞(see section 3.2).

4.2.2 Saddle-Node bifurcation cascades with different basic period

For each basic period there will be one saddle-node bifurcation cascade, and therefore a family of functions $g_{i,q}$. Following the explanation of section 4.2.1 there will be an equation like Eq. (6) for each bifurcation cascade. All equations will be identical and all of them will approach to r_∞.

We have already found this result in section 3 (see Eq. (8)).

4.3 Extension to non-canonical windows

Until now we have only been working in the canonical window, but it is not difficult to extend the results to any window. For instance, if we had worked in a period-p window then a $q \cdot 2^n$ saddle-node bifurcation cascade would have occurred inside it. It would be enough to take into account that the $R_{n,q}$ of Eq. (17) is the value for which the supercycle associated to a saddle-node period-$q * 2^n$ orbit occurs, but this time inside a period-p window, so that the whole process is identical and the same universal results are obtained. The difference would lie in the fact that the approaching to the Eq. (20) would be at a Myrberg-Feigenbaum point $r_{\infty,p}$ of a period-p window instead of at the Myrberg-Feigenbaum point r_∞ of the canonical window. Therefore, if r_∞ were

substituted by $r_{\infty,p}$ all equations valid in the canonical window would remain valid in a period-p window.

4.4 Extension to functions with a non-quadratic maximum

As we have generalized the results from a canonical window to an arbitrary periodic window, one more generalization remains to be done: the results are generalized to maps other than logistic ones.

In this paper, the work has been developed with Eq. (1), which is topologically conjugate to

$$x_{n+1} = x_n^2 - c$$

The whole work could have been developed with the function

$$f(x, c) = x^2 - c$$

and it could have been repeated in an identical way with the family of functions

$$f_n(x, c) = x^{2n} - c \quad n = 1, 2, 3, ... \tag{26}$$

The sections 4.1 and 4.2 can be rewritten, step by step, defining a new family of functions $g_{i,q}$, which are identical to the ones given by Eq. (17), but this time f is replaced by Eq. (26). This new family of functions applies to an equation identical to Eq. (18), and it results in Eq. (20), which has a solution [18], once the maximum of $g(x)$has been fixed as x^{2n} , $n = 1, 2, 3, ...$, although α and δ will be different, depending on the kind of maximum.

As the universal behavior comes from Eq. (20), and this remains identical, it turns out that $g^q$is self-similar, although we will have a different $g(x)$ depending on the kind of maximum we choose. We will see how the graph of functions x^{2n} is repeated in the saddle-node bifurcation cascade.

The results relative to saddle-node bifurcation cascades remain the same, because these results came from Eqs. (25), (23) and (24), which did not change after f was replaced by Eq. (26) in $g_{i,q}$. As everything is identical it turns out that equations identical to Eq. (6) will be obtained, but the value of δ will depend on the kind of maximum with which we are dealing with.

In short, in this section we have expound the convergence to Feigenbaum constant α and the self-similarity of the function f^q in the neighborhood of

$\left(\frac{1}{2}, \frac{1}{2}\right)$ as it was shown in section 2. We have also laid the foundations for a generalization of the Feigenbaum Eq. (6) and, at last, we have generalized these results to functions with maximum like x^{2n} , $n = 1, 2, 3, ...$

5 Generalization of Bifurcation Rigidity Principle

We have reached Eq. (6) (see section 3.1) coming from the bifurcation rigidity Principle [14] for functions with a quadratic maximum. The proof can be followed the other way round, starting from Eq. (6) and reaching the bifurcation rigidity Principle. If we start from Eq. (6), valid for functions like Eq. (26), we will arrive to the bifurcation rigidity Principle. This time it will be valid for functions like Eq. (26) and not only for functions with quadratic maximum as it was initialed expounded.

Let a function be like Eq. (26). According to section 4.4, we know that both successive pitchfork and successive saddle-node bifurcations are governed by an equation like

$$r_{n+1} = \frac{1}{\delta} r_n + (1 - \frac{1}{\delta}) r_\infty \tag{27}$$

In period-j window successive saddle-node bifurcations will be governed by

$$r_{q*2^{n+1},SN,j} = \frac{1}{\delta} r_{q*2^n,SN,j} + (1 - \frac{1}{\delta}) r_{\infty,j} \tag{28}$$

meanwhile in a canonical window the successive saddle-node bifurcations will be governed by

$$r_{q*2^{n+1},SN} = \frac{1}{\delta} r_{q*2^n,SN} + (1 - \frac{1}{\delta}) r_\infty \tag{29}$$

Accordingly there is a linear transformation between both saddle-node bifurcation cascades given by

$$\frac{r_{q*2^{n+1},SN,j} - \frac{1}{\delta} r_{q*2^n,SN,j}}{r_{q*2^{n+1},SN} - \frac{1}{\delta} r_{q*2^n,SN}} = \frac{r_{\infty,j}}{r_\infty} \tag{30}$$

For pitchfork bifurcations there is an identical equation

$$\frac{r_{n+1,j} - \frac{1}{\delta} r_{n,j}}{r_{n+1} - \frac{1}{\delta} r_n} = \frac{r_{\infty,j}}{r_\infty} \tag{31}$$

where n is related to n-th pitchfork bifurcation.

Let's take the intervals

$[r_{n,j}; r_{q*2^n,SN,j}]$ and $[r_{n+1,j}; r_{q*2^{n+1},SN,j}]$
in the period-j window and the intervals

$[r_n; r_{q*2^n,SN}]$ and $[r_{n+1}; r_{q*2^{n+1},SN}]$
in the canonical window. If a linear transformation maps the interval $[r_{n,j}; r_{q*2^n,SN,j}]$ onto the interval $[r_n; r_{q*2^n,SN}]$then Eqs. (30) and (31) will force the interval $[r_{n+1,j}; r_{q*2^{n+1},SN,j}]$ to be linearly transformed into the $[r_{n+1}; r_{q*2^{n+1},SN}]$. Accordingly the bifurcation diagrams of both window are linearly changed as set by the bifurcation rigidity Principle, but this time the Principle works with x^{2n}-maps and not only with quadratic maps. As the Eq. (27) is applied to $n \gg 1$, the smaller the windows are, the better the linear approximation to the transformation will be.

6 Symbolic sequences and orbit ordering

Sequence principle.- Let be a period-$p*(q*2^n)$ saddle-node orbit. The points of this orbit lie inside the $p*2^n$ neighborhoods of the $p*2^n$ supercycle points of f^{p*2^n}, each one of these neighborhoods having q saddle-node points. Then the sequence of the saddle-node orbit is obtained with the following process

i) Write the sequence of the orbit of the supercycle of f^{p*2^n}, which will be like $CI_1.....I_{p*2^n-1}$, where C indicates the center and represents the point $x = \frac{1}{2}$ and I_i , $i = 1, ..., p*2^n-1$ can take the values L (left) or R (right) with respect to C.

ii) Write consecutively q times the sequence obtained in the former point i), getting a sequence like

$C_1I_1.....I_{p*2^n-1}C_2I_1.....I_{p*2^n-1}.........C_qI_1.....I_{p*2^n-1}$.

iii) Write the sequence of period-q saddle-node orbit, that is, the sequence of the saddle-node orbit of f^q.

iv) Calculate $(-1)^n$. If the result is negative then conjugate the letters obtained in point iii) by means of L↔R. Bear in mind that $n \in Z^+$.

v) Replace consecutively each letter $C_1C_2.....C_q$ of the sequence obtained in ii) by the complete sequence obtained in iv), keeping the order.

Proof

i) Let be period-$p*2^n$ supercycle at $r = r_0$, in the period-p window, whose points are $\{x_1, ...x_{p*2^n}\}$. The order in which these points are visited determines the sequence of the supercycle orbit of f^{p*2^n}, that is, $CI_1.....I_{p*2^n-1}$.

ii) The function f^{p*2^n}can be approximated by a quadratic polynomial $g(x)$ in the neighborhood of each point of the supercycle $\{x_1, ...x_{p*2^n}\}$ [14].

The period-$p*(q*2^n)$ saddle-node orbit of f occurs when $g(x)$ has a period-q saddle-node orbit. This period-q saddle-node orbit is like $\{x_0, g(x_0), ..., g^{q-1}(x_0)\}$, with $g^q(x_0) = x_0$. As $g(x)$ is an approximation of $f^{p*2^n}$to pass from point $g^i(x_0)$ to point $g^{i+1}(x_0)$ we will have to iterate $p*2^n$ times the function f, that is, the iterated of f visit the neighborhood of the points of the supercycle $\{x_1, ...x_{p*2^n}\}$. With each $p*2^n$iterated, the same neighborhood is visited again, not to the same point but the following one of the period-q saddle-node orbit trapped in this neighborhood.

With each $p*2^n$ iterated of f we go back to the same neighborhood of a given point of the supercycle. This neighborhood will be to the left or to the right of the center C, or it will be the point C itself. For the time being, let's leave the neighborhood of the point C and let's consider the period-q saddle-node orbits trapped in the other neighborhoods. If this orbit is trapped in a neighborhood to the left (right) of the center C then all its points will be placed to the left (right) of C and they will generate an L (R) in the sequence of the period-$p*(q*2^n)$ saddle-node orbit. Therefore, writing consecutively q times the sequence of the orbit of the supercycle we would obtain the sequence of the period-$p*(q*2^n)$ orbit.

iii) A drawback of the previous procedure takes place when the neighborhood of the point C is visited. For in this neighborhood not all the points of the period-q saddle-node orbit are to the left or to the right of the point C. Apparently, to solve the problem, it would be enough to calculate the sequence of the period-q saddle-node orbit, and with this sequence to replace the $C_1C_2.....C_q$ of the sequence $C_1I_1.....I_{p*2^n-1}C_2I_1.....I_{p*2^n-1}.........C_qI_1.....I_{p*2^n-1}$. This is so because each time the neighborhood of C is visited a point of the period-q saddle-node orbit trapped in this neighborhood is visited.

iv) However there is a drawback as the letters $C_1C_2.....C_q$are replaced by the the letters of period-q saddle-node orbit, because the period-q saddle-node orbit is conjugated in the neighborhood of C, as the value of n is in f^{p*2^n}. This is so because of the negative sign in Eq. (21). If n is even the sequence holds and if n is odd the sequence is conjugated. Therefore to solve the problem $(-1)^n$ is calculated and if this value is negative the sequence of the orbit is conjugated.

v) Finally, it is enough to replace the letters $C_1C_2.....C_q$ by the letters of the period-q saddle-node orbit sequence correctly conjugated to obtain the right

result.

Notice that it is enough to know the sequences of the supercycles of f^{2^n} and the primary saddle-node orbits f^p to obtain the sequence of each orbit in the logistic map. According to the rule that we show here we get the sequence of the supercycle of f^{p*2^n} in the period-p window, and from this supercycle we calculate the sequence of the period-$p*(q*2^n)$ saddle-node orbit that will occur in a period-q window inside a period-p window.

Let's see how the principle is used to obtain the period-$3*2$ saddle-node orbit sequence of Fig. 2, in which $p=1, n=1, q=3$.

i) This time there is a supercycle of f^2, whose orbit is formed by the points $\left\{\frac{1}{2}, f(\frac{1}{2})\right\}$ and its own sequence is CR.

ii) There is a period-3 saddle-node orbit which is trapped in the neighborhoods of $\frac{1}{2}$ and $f(\frac{1}{2})$, accordingly we write $CRCRCR$

iii) The sequence of the period-3 saddle-node is RRL, as shown in Fig. 1

iv) As $(-1)^1=-1$ the sequence RRL is conjugated to give LLR

v) Finally, the letters C of the sequence $CRCRCR$ are replaced by the letters of the orbit LLR, turning into $LRLRRR$. The first letter indicates a place with respect to C, the point $x=\frac{1}{2}$. In Fig. 2 it can be observed that the saddle-node orbit goes through as the sequence obtained .

The principle of ordering shown above, for saddle-node orbits, is very similar to the one described by Derrida *et al.* in [19] for supercycle orbits.

Principle of saddle-node orbits ordering.- Let be f the logistic map, if $q>s$ then the period-$qp2^n$saddle-node orbit occurs at a value r smaller than that for which the period-$sp2^n$saddle-node orbit occurs, that is, the period $qp2^n$ precedes period $sp2^n$ in the parameter space. We note it down like $qp2^n \triangleright sp2^n$ if $q>s$.

Proof

As the function f^{p*2^n}can be approximated by a quadratic polynomial $g(x)$ in the neighborhood of each point of the supercycle $\{x_1, ...x_{p*2^n}\}$ [14], it will turn out that as $g(x)$ has a period-s saddle-node orbit then f will have a period-$sp2^n$ orbit , and as $g(x)$ has a period-q saddle-node orbit then f will have a period-$qp2^n$ orbit. Since for the saddle-node $q \triangleright s$ if $q>s$ so $qp2^n \triangleright sp2^n$ if $q>s$.

Comments:

i) Although we have a Sharkovsky ordering the principle is not limited to first-occurrence orbits to which the Sharkovsky theorem applies.

ii) We cannot discard halfway orbits in this ordering, which come from other values of n, p or q, that is, we have ordered non-first-occurrence orbits althugh all of them.

7 Conclusion

The Feigenbaum equation

$$r_{n+1} = \frac{1}{\delta} r_n + (1 - \frac{1}{\delta}) r_\infty$$

which allows us to locate the successive pitchfork bifurcations in a period doubling cascade, turns out to be an expression of a general nature. This is applied to cascades of other objects present in the logistic equation. For instance, this expression rules, among other things, saddle-node bifurcations, Myrberg-Feigenbaum points, and the birth and death of some windows. That is, not only the equation gives the ordering but also the parameter values for which such objects appear.

An immediate consequence of these results is the posibility to determine: the scaling of windows, the parameters of birth and death, the relative position of the Myrberg-Feigenbaum points, the relative position of the attractors in which the period doubling cascade finishes.

We would like to highlight another consequence of these results, a new concept: an attractor of attractors, which coincide with the Myrberg-Feigenbaum points.

The former results have been first expounded with linear transformations benefiting from the properties of the windows (bifurcation rigidity Principle), and later, part of theses results have been derived using the linearization of the Feigenbaum-Cvitanoviç equation. Later the bifurcation rigidity Principle is generalized to functions with a maximum other than quadratic.

The concept upon which the whole work relies is the saddle-node bifurcation cascade, which is a sequence of successive saddle-node bifurcations, which in turn double the number of saddle-node fixed points. The bifurcation points are ruled by the same equation obtained by Feigenbaum for the period doubling cascade. By using the bifurcation parameter given by saddle-node bifurcation cascade, we follow the Feigenbaum work to end up in a Feigenbaum-Cvitanoviç

equation, which proves the universal behavior of our results, and it allows us the generalization to maps other than the logistic one.

The saddle-node bifurcation cascade shows a further behavior similar to a period doubling cascade: it is self-similar after successive iterations. Nonetheless, for a fixed period that is duplicated in each bifurcation, there are different saddle-node bifurcations, however this they have the same accumulating point and the same scaling law.

The ordering of orbits missing from Sharkovsky theorem have been given, as well as the ordering of Myrberg-Feigenbaum points associated to the windows born of saddle-node bifurcation cascades.

Lastly symbolic sequences of the orbits found in the saddle-node bifurcation cascade have been described.

We have summarized above the mathematical contributions of our work. We started the introduction by pointing out that any progress, in the logistic equation, would also yield progress to Physics . We would like to quote "**the self-similar cascade of bifurcations** (....) **it is characterized by an infinite series of saddle-node bifurcation of cycles, accumulating at a finite parameter value**" reported by Yeung and Strogatz [20]. This mechanism coincides with the one described for a saddle-node bifurcation cascade, where the accumulating parameter value is the Myrberg-Feigenbaum point, which will most likely make the phenomenon fit in the framework described in this work.

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Table 1

Period $3 \cdot 2^n$ Saddle-Node bifurcation cascade inside the canonical window. The convergence to α and δ Feigenbaum constants is shown.

Period	r_{3*2^n}	δ_{3*2^n}	α_{3*2^n}	n
3	3.82842712			0
$3*2$	3.62655316		2.30066966	1
$3*2^2$	3.58202300	4.5334294	2.52112183	2
$3*2^3$	3.57253281	4.69223061	2.49753642	3
$3*2^4$	3.57049971	4.66784221	2.50264481	4
$3*2^5$	3.57006433	4.66971381	2.50263906	5

Table 2

Period $5 \cdot 2^n$ Saddle-Node bifurcation cascade inside the canonical window. The convergence to Feigenbaum δ constant is shown.

Period	r_{5*2^n}	δ_{5*2^n}	n
5	3.73817238		0
$5*2$	3.60520807		1
$5*2^2$	3.57751225	4.80088006	2
$5*2^3$	3.57156559	4.65737406	3
$5*2^4$	3.57029262	4.67144810	4
$5*2^5$	3.57001998	4.66925870	5

Table 3

Period $3 \cdot 2^n$ Saddle-Node bifurcation cascade inside the period-3 window. The convergence to Feigenbaum δ constants is shown.

Period	r_{3*2^n}	δ_{53*2^n}	n
3	3.85361311		0
$3*2$	3.85031470		1
$3*2^2$	3.84962047	4.75117756	2
$3*2^3$	3.84947362	4.72747702	3
$3*2^4$	3.84944223	4.67824148	4
$3*2^5$	3.84943551	4.67113095	5

Table 4
Period $3 \cdot 2^n$ Saddle-Node bifurcation cascade inside the period-5 window. The convergence to Feigenbaum δ constants is shown.

Period	r_{3*2^n}	δ_{3*2^n}	n
3	3.744003		0
$3*2$	3.74321655		1
$3*2^2$	3.743050198	4.727522	2
$3*2^3$	3.7430150831	4.737633	3
$3*2^4$	3.7430075758	4.677634	4
$3*2^5$	3.7430059688	4.671437	5

Fig. 1. Period-3 Saddle-Node orbit in the canonical window, for $r \simeq 3.82842712$.

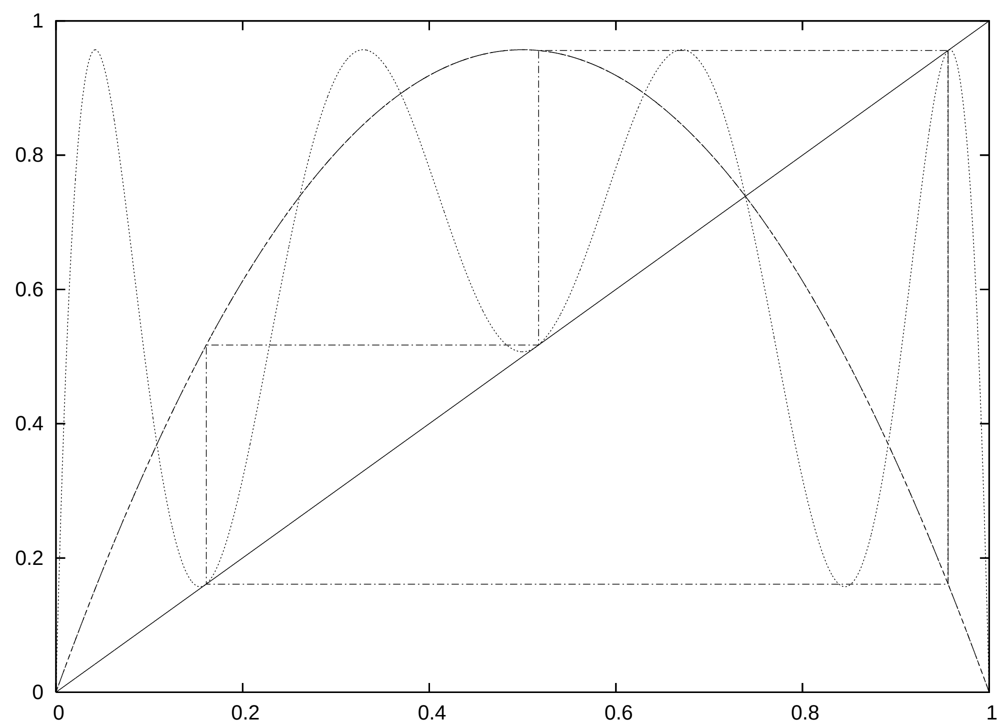


Fig. 2. Period-$3 \cdot 2$ Saddle-Node orbit in the canonical window, for $r \simeq 3.62655316$.

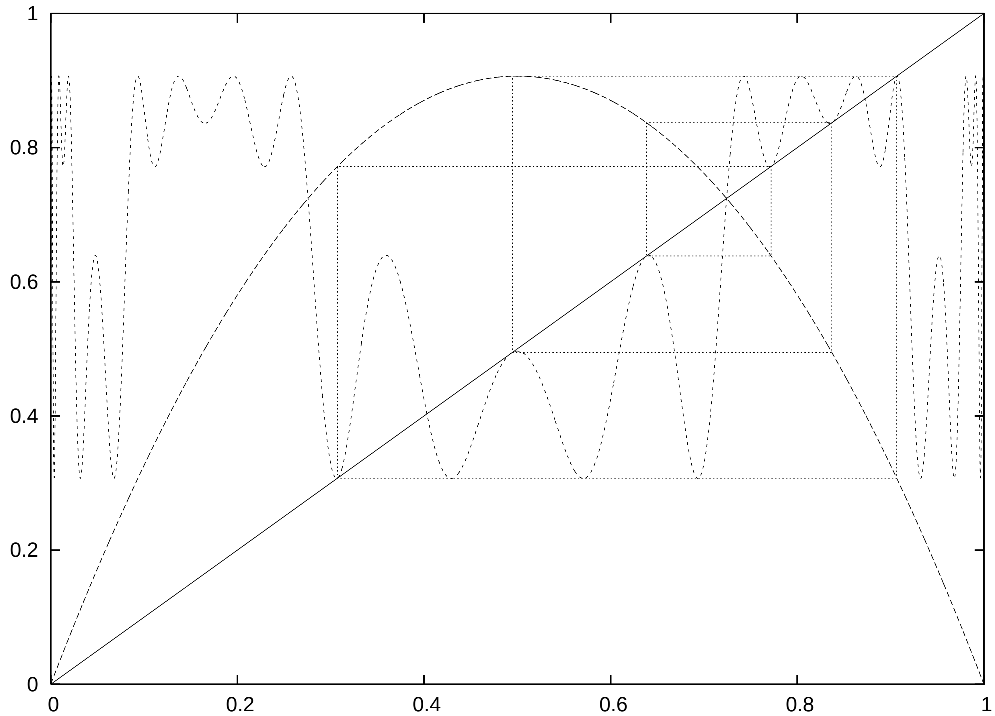


Fig. 3. Period-$3 \cdot 2^2$ Saddle-Node orbit in the canonical window, for $r \simeq 3.58202300$.

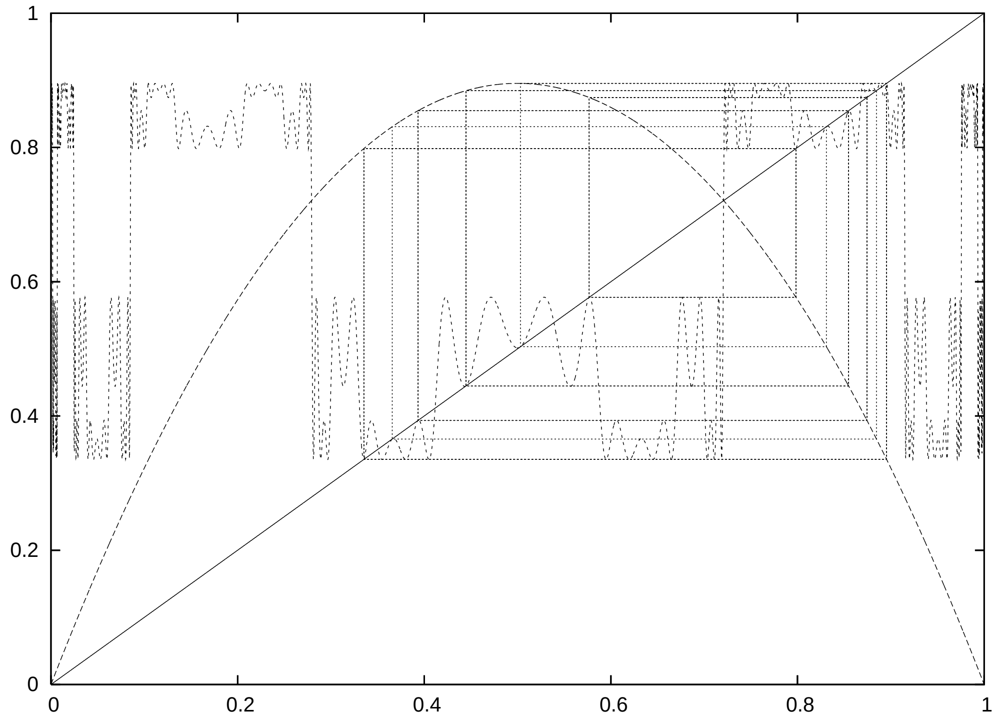


Fig. 4. Period-3 Saddle-Node orbit in the period-5 window, for $r \simeq 3.74400300$. The shape of Fig. 1 is reproduced five times; here is shown a magnification of the replica near (1/2, 1/2).

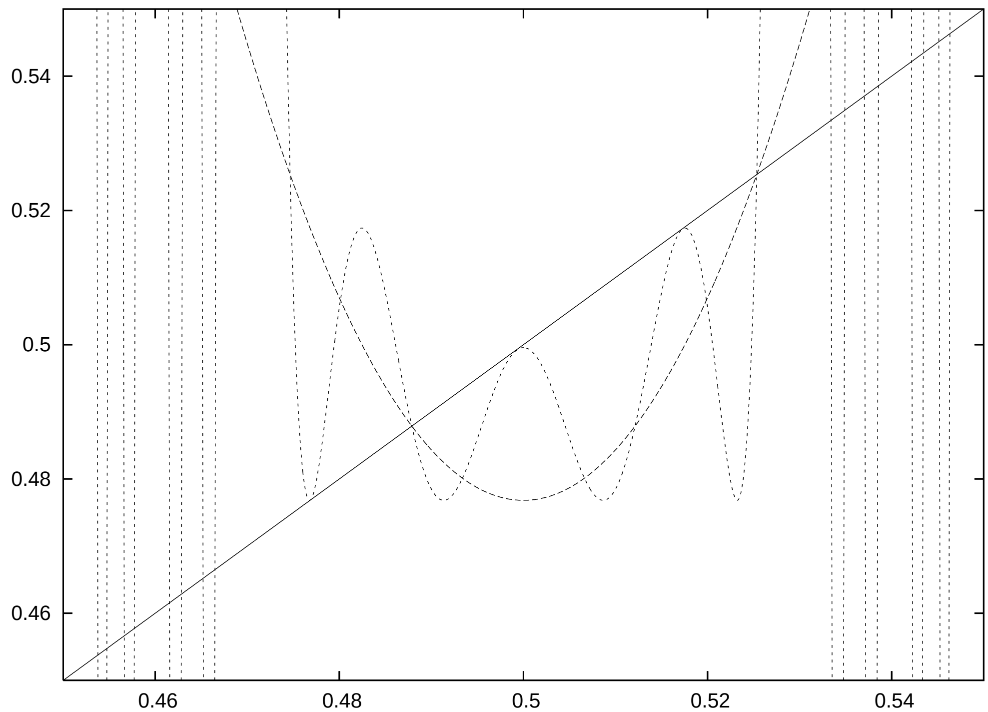


Fig. 5. Period-$3 \cdot 2$ Saddle-Node orbit in the period-5 window, for $r \simeq 3.74321655$. The shape of Fig. 2 is reproduced five times; here is shown a magnification of the replica near (1/2, 1/2).

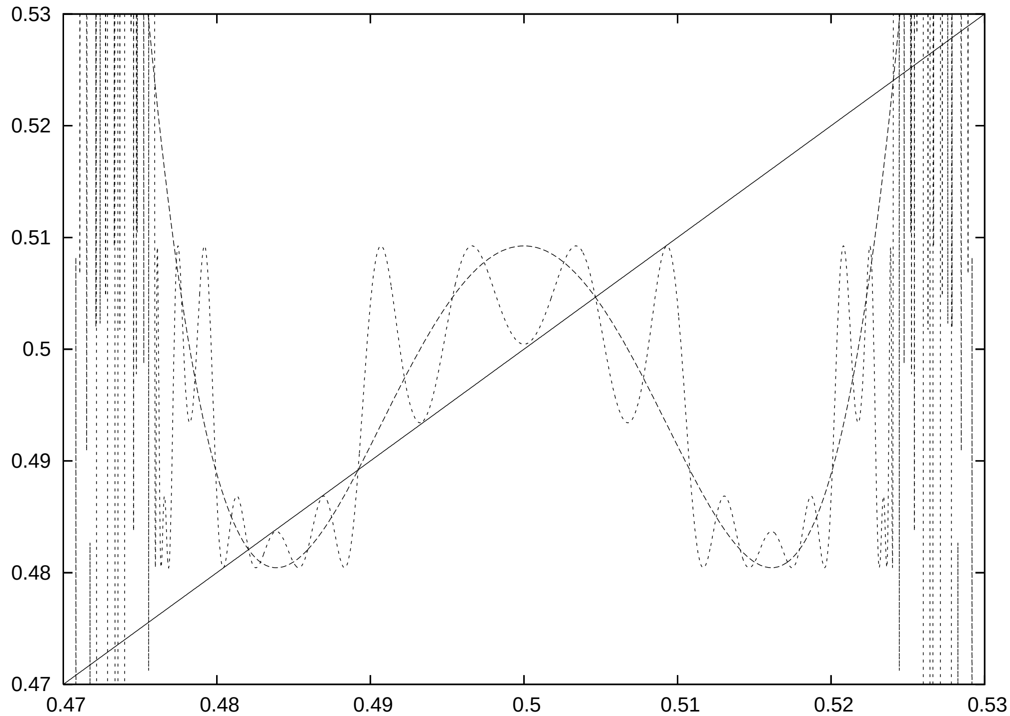


Fig. 6. One of the three possible period-6 Saddle-Node orbits close to $r = 3.97790000$.

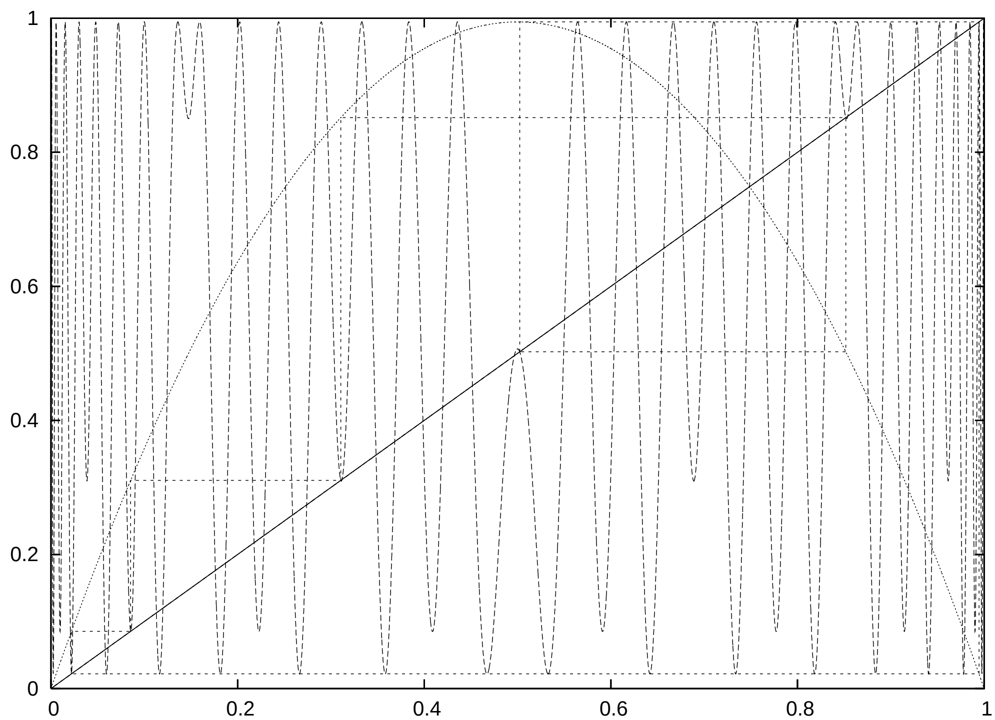
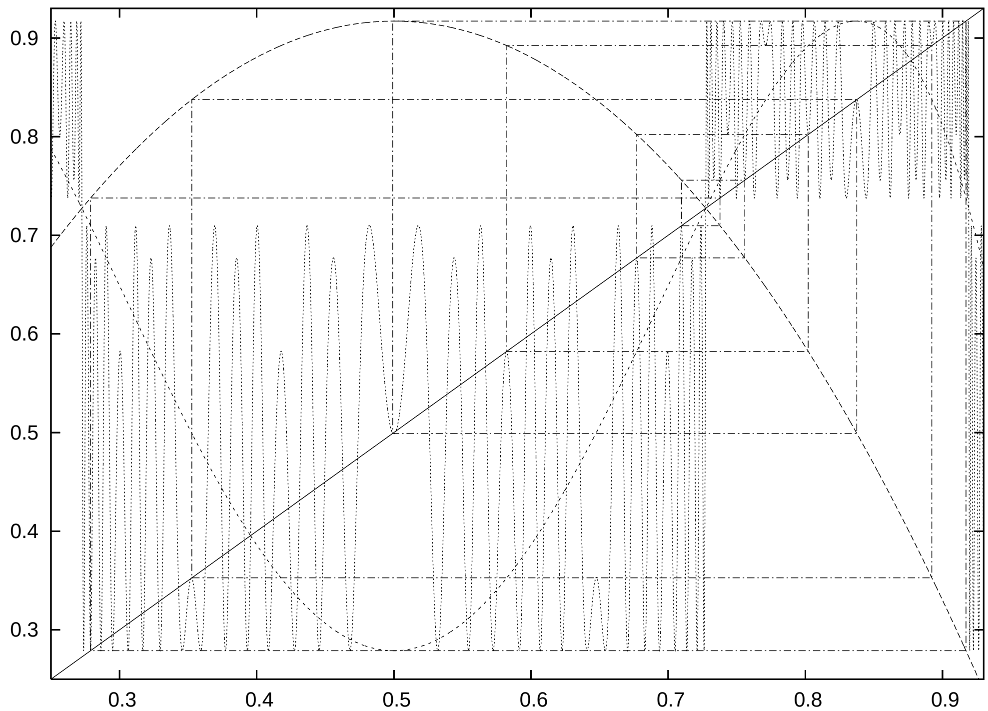


Fig. 7. Period-6 · 2 Saddle-Node orbit in the canonical window, close to $r \simeq 3.66840000$. The shape of Fig. 6 is reproduced twice.



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